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Comonotone Approximation by Splines of Piecewise Monotone Functions

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INTRODUCTION

In 1977 DeVore [5] obtained Jackson-type estimates for the approximation of monotone functions by monotone splines. Since then various authors [1, 4, 7] have considered extensions of the problem and improvements of the results or the methods of proof. Recently Beatson [3] has introduced an economical method of estimating the rate of monotone approximation by splines by means of similar estimates on the rate of restricted range approximation.

The purpose of this work is to obtain Jackson-type estimates for the approximation of a piecewise monotone function f; i.e., f changes its monotonicity finitely many times in the interval, by means of splines which are comonotone with it. We apply some of the methods that were introduced by Beatson [3] and obtain estimates that involve the modulus of continuity of the highest continuous derivative that f possesses.

It is interesting to compare these estimates with what is known for comonotone approximation of piecewise monotone functions by polynomials. In this case the only estimates known involve the modulus of continuity of the function itself. These have been obtained independently by Iliev [6] and Newman [8] and no progress has been made since then.

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The Main Results

We consider continuous functions on $[a, b]$ that are piecewise monotone, i.e., change direction from increasing to decreasing or vice versa finitely many times, say, $r \geqslant 1$, in $[a, b]$.

Let $T: a = t_0 < t_1 < \cdots < t_n = b$ be a mesh on $[a, b]$ and for $k \geqslant 1$ let $\mathscr{S}(k, T)$ denote the space of splines of order k with knots T, i.e., $s \in \mathscr{S}(k, T)$ if and only if $s \in C^{(k-2)}[a, b]$ and s is a polynomial of degree $\leqslant k-1$ on each interval $[t_{i-1}, t_i]$, $i = 1,..., n$. Sometimes in order to relate splines in adjacent intervals we will take for T an infinite sequence of strictly increasing knots $\{t_i\}_{i=-\infty}^{\infty}$ and $\mathscr{S}(k, T)$ will mean the obvious (modified) space of splines on $(\inf t_i, \sup t_i)$. Back to our original mesh on $[a, b]$, if $\delta = \max_{1 \leqslant i \leqslant n} (t_i - t_{i-1})$ denotes the mesh size and if $\omega(f, \cdot)$ denotes the usual modulus of continuity of f, then we prove the following

THEOREM. *For $k \geqslant 1$ there is a constant $C = C(k, r)$ depending only on k and r such that if $f \in C^j[a, b]$ for some $0 \leqslant j \leqslant k-1$ is a piecewise monotonic function with r turning points, then there exists an $s \in \mathscr{S}(k, T)$ comonotone with f such that*

$$\|f - s\| \leqslant C\delta^j \omega(f^{(j)}, \delta). \tag{1}$$

For the proof we need two auxiliary lemmas due to Beatson [2].

LEMMA 1. *For $j \geqslant 0$ let π_j denote the space of polynomials of degree $\leqslant j$. Let $l(x)$ and $u(x)$ be two extended real valued functions on $[a, b]$ and set*

$$W = \{g \in C[a, b]: l(x) \leqslant g(x) \leqslant u(x), a \leqslant x \leqslant b\}.$$

Suppose that $f \in C^j[a, b] \cap W$ and that $\pi_j \cap W$ is not empty. Then there exists a polynomial $p \in \pi_j \cap W$ such that

$$\|f - p\| \leqslant (b-a)^j \omega(f^{(j)}, b-a). \tag{2}$$

LEMMA 2. *Let $k \geqslant 2$ be an integer and $d = 2(k-1)^2$, and let T: $\{t_i\}_{i=-\infty}^{\infty}$ be a strictly increasing knot sequence with $t_0 = a$ and $t_d = b$. If p_1 and p_2 are two polynomials of degree $\leqslant k-1$, then there exists a spline $s \in \mathscr{S}(k, T)$ such that $s(x)$ is a number between $p_1(x)$ and $p_2(x)$ for each $x \in [a, b]$ and*

$$s(x) = p_1(x) \text{ on } (-\infty, a], \qquad s(x) = p_2(x) \text{ on } [b, \infty).$$

Proof of the Theorem. The main part of the proof will involve a construction of a suitable spline approximating the derivative of f and then integration. Therefore we will first assume that $j \geqslant 1$ and will deal with the case $j = 0$ at the end. Let $n > 2d$ (where $d = 2(k-1)^2$) and define the points $x_i = t_{id}, i = 0, 1, ..., m,$ where $m = [n/d]$ and $x_{m+1} = t_n$. Now let $I_i = [x_i, x_{i+2}]$, $i = 0, ..., m-1$. If f' is of fixed sign on I_i let the polynomials $P_{li}, P_{ui} \in \pi_{j-1}$ satisfy

$$\begin{aligned} 0 \leqslant P_{li} \leqslant f' \leqslant P_{ui} \quad &\text{on} \quad I_i \quad \text{if} \quad f' \geqslant 0 \quad \text{on} \quad I_i, \\ P_{li} \leqslant f' \leqslant P_{ui} \leqslant 0 \quad &\text{on} \quad I_i \quad \text{if} \quad f' \leqslant 0 \quad \text{on} \quad I_i, \end{aligned} \tag{3}$$

and

$$\max\{\|f' - P_{li}\|_{I_i}, \|f' - P_{ui}\|_{I_i}\} \leqslant (2d\delta)^{j-1} \omega(f^{(j)}, \delta) = \varepsilon, \tag{4}$$

for example. Also for each collection $I_{\mu\nu} = \bigcup_{i=\mu}^{\nu} I_i, 0 \leqslant \mu \leqslant \nu \leqslant m-1$, define the polynomial $P_{\mu\nu} \in \pi_{j-1}$ such that

$$\min(f', 0) \leqslant P_{\mu\nu} \leqslant \max(f', 0) \qquad \text{on} \quad I_{\mu\nu} \tag{5}$$

and

$$\|f' - P_{\mu\nu}\|_{I_{\mu\nu}} \leqslant (\nu - \mu + 1)^{j-1} \varepsilon. \tag{6}$$

The existence of these polynomials is guaranteed by Lemma 1. (In all three cases the function f' to which we apply the lemma and the polynomial $p \equiv 0$ belong to the respective W as required.)

We divide the intervals I_i, $1 \leqslant i \leqslant m-1$, into intervals of two types. The interval I_i is of type I if $i = 0$ or if f' changes sign (at least once) in $I_{i-1} \cup I_{i+1}$. Otherwise it is of type II. Employing an idea of Beatson [2] we now define the required spline $s \in \mathscr{S}(k, T)$ inductively starting with $i = 0$. Let $I_\mu, \mu \geqslant 1$, be the first interval of type II. (If there is none, then $\mu = m+1$.) Then define $s' = P_{o\mu}$ on $[a, x_\mu]$ and $s(a) = f(a)$, and define $g_{\mu-1} = P_{o\mu}$ on $I_{\mu-1}$. Suppose that $s(x)$ is defined on $[a, x_\nu]$ $\nu \leqslant m$ and the polynomial $g_{\nu-1}$ is defined on $I_{\nu-1}$, let us extend s onto $[x_\nu, x_{\nu+1}]$ (or onto a bigger interval) and get another polynomial g. If $\nu = m$ define $s' = g_{m-1}$ on $[x_m, x_{m+1}]$. Assume $\nu < m$. If I_ν is of type II then f' has constant sign there. On I_ν define $g_\nu = P_{u\nu}$ if $f(x_\nu) \geqslant s(x_\nu)$, and $g_\nu = P_{l\nu}$ if $f(x_\nu) \leqslant s(x_\nu)$. Now define s' on $[x_\nu, x_{\nu+1}]$ as the smoothing spline between $g_{\nu-1}$ and g_ν which is obtained from Lemma 2. If on the other hand I_ν is of type I, then let I_ρ be the first interval of type II. (Again if there is none, then $\rho = m+1$.) Define $g_\nu = P_{\nu\rho}$ on I_ν and $g_{\rho-1} = P_{\nu\rho}$ on $I_{\rho-1}$. Now define s' on $[x_\nu, x_{\nu+1}]$ as the smoothing spline between $g_{\nu-1}$ and g_ν and $s' = P_{\nu\rho}$ on $[x_{\nu+1}, x_\rho]$. Finally $s(x) = s(a) + \int_a^x s'(t)\,dt$ is a spline in $\mathscr{S}(k, T)$ which is comonotone with f and we only have to show that the error estimate is valid. To this end we observe that since there are at most $3r+1$ intervals of type I (the additional

1 is due to the fact that I_0 is treated as being of type I in any case) then it follows by (4) and (6) that on $[a, b]$

$$\|f' - s'\| \leqslant (3r+1)^{j-1}\,\varepsilon.$$

Hence if $E = f - s$ is the error function, then

$$\|E'\|_{[a,b]} \leqslant A(k,r)\,\varepsilon,$$

where $A = (3r+1)^{k-1}$. Thus for $x, y \in [x_i, x_{i+1}]$ we have

$$|E(x) - E(y)| \leqslant A\,d\delta\varepsilon. \tag{8}$$

Since there are at most $3r+1$ intervals of type I, $3r$ of which come in overlapping triples, the total variation of E on these intervals is bounded by $2A(2r+1)\,d\delta\varepsilon$. On the other hand if I_ν is an interval of type II and I_μ is the closest interval of type I to the left of I_ν (here too I_0 is treated as if it is of type I), then we will show that

$$|E(x)| \leqslant |E(x_\mu)| + 2A\,d\delta\varepsilon, \qquad x \in [x_\nu, x_{\nu+1}]. \tag{9}$$

This, in addition to the fact that $E(a) = 0$, yields, for all $x \in [a, b]$,

$$\begin{aligned} |E(x)| &\leqslant 4A(r+1)\,d\delta\varepsilon \\ &\leqslant C\delta\varepsilon \\ &= C\delta^j\,\omega(f^{(j)}\delta) \end{aligned}$$

and (1) is evident with $C = (3r+1)^{k-1}(r+1)\,2^{k+1}\,d^k$.

In order to prove (9) observe that if $|E(x_\nu)| \leqslant |E(x_\mu)| + Ad\delta\varepsilon$ then (9) follows by (8). Thus assume $|E(x_\nu)| > |E(x_\mu)| + Ad\delta\varepsilon$, which in turn implies that $\nu \geqslant \mu + 2$. This means in particular that $\nu - 1 > \mu$ so that $I_{\nu-1}$ is also of type II. By virtue of (8)

$$E(x)\,E(x_\nu) > 0, \qquad x \in I_{\nu-1}, \tag{10}$$

and the construction of the polynomials $g_{\nu-1}$ and g_ν imply that $(f'(x) - g_{\nu-1}(x))\,E(x_{\nu-1}) \leqslant 0$ on $I_{\nu-1}$ and $(f'(x) - g_\nu(x))\,E(x_\nu) \leqslant 0$ on I_ν. Now by (10) $g_{\nu-1}(x)$ and $g_\nu(x)$ are on the same side of $f'(x)$ for $x \in [x_\nu, x_{\nu+1}]$, hence so is $s'(x)$. Since $E(x) \neq 0$ on $[x_\nu, x_{\nu+1}]$ (again by (10)), it follows that $|E(x)|$ is nonincreasing in $[x_\nu, x_{\nu+1}]$. The argument applied here is valid for any interval $I_\rho, \mu < \rho < \nu$ where

$$|E(x_\rho)| > |E(x_\mu)| + Ad\delta\varepsilon,$$

so we work our way from μ to ν. If ρ is the first index for which this inequality holds, then

$$|E(x_{\rho-1})| \leqslant |E(x_\mu)| + Ad\delta\varepsilon$$

and by virtue of (8)

$$|E(x_\rho)| \leqslant |E(x_\mu)| + 2Ad\delta\varepsilon.$$

Also, by the discussion above, $|E(x)|$ is nonincreasing in $[x_\rho, x_{\rho+1}]$ so that (9) holds there. If $|E(x_{\rho+1})| > |E(x_\mu)| + Ad\delta\varepsilon$ we continue the downward trend so that (9) holds in $[x_{\rho+1}, x_{\rho+2}]$ and if $|E(x_{\rho+1})| \leqslant |E(x_\mu)| + Ad\delta\varepsilon$ then (9) holds in $[x_{\rho+1}, x_{\rho+2}]$ by virtue of (8). Finitely many steps bring us to $[x_\nu, x_{\nu+1}]$.

If $n \leqslant 2d$, let $P \in \pi_{j-1}$ satisfy

$$\min(f', 0) \leqslant P \leqslant \max(f', 0) \qquad \text{on } [a, b],$$
$$\|f' - P\| \leqslant (b-a)^{j-1}\, \omega(f^{(j)}, b-a).$$

Such P is guaranteed by Lemma 1. Define $s(x) = f(a) + \int_a^x P(t)\, dt$, then s is comonotone with f on $[a, b]$ and

$$\|f - s\| \leqslant (b-a)^j\, \omega(f^{(j)}, b-a). \tag{11}$$

Since $b - a \leqslant n\delta \leqslant 2d\delta$, (11) implies (1) with $C = (2d)^{k+1}$ and our proof for $j \geqslant 1$ is complete. The case $k - 1 = j = 0$ is trivial.

We conclude with the case $k - 1 > j = 0$. Given $f \in C[a, b]$ having r turning points in $[a, b]$ we will construct a $\psi \in C^1[a, b]$ having the same turning points which is close enough to f, and approximate ψ by our construction above ($j = 1$). Given δ divide the interval $[a, b]$ into $n + 1$ equal intervals each of length $\leqslant \delta$, where $n = [(b-a)/\delta]$. Then in each subinterval (a', b') where f changes direction at least once or which is adjacent to an interval where f changes direction, set $h(x) = 0$. Otherwise set $h(x) = f(b') - f(a')/b' - a'$. Define $g(x) = f(a) + \int_a^x h(t)\, dt$, then it is easy to see that

$$\|f - g\| \leqslant (3r + 1)\, \omega(f, \delta) \tag{12}$$

and

$$\|g'\| \leqslant (2/\delta)\, \omega(f, \delta). \tag{13}$$

Now define the Steklov transform $\psi(x) = (1/\delta) \int_{x-\delta/2}^{x+\delta/2} g(t)\, dt$. By the construction of g it follows that ψ is comonotone with f. Also by (12)

$$\begin{aligned} \|f - \psi\| &\leqslant \|f - g\| + \omega(f, \delta) \\ &\leqslant (3r + 2)\, \omega(f, \delta). \end{aligned} \tag{14}$$

Since $\psi \in C^1[a, b]$ we may apply what we have already proved. Thus there is a spline $s \in \mathscr{S}(k, T)$ comonotone with ψ and therefore with f such that

$$\|s - \psi\| \leqslant C\delta\omega(\psi', \delta). \tag{15}$$

Now

$$\psi'(x) = (1/\delta)[g(x + \delta/2) - g(x - \delta/2)]$$

so that

$$\begin{aligned} \|\psi'\| &\leqslant (1/\delta)\,\omega(g, \delta) \\ &\leqslant \|g'\|. \end{aligned} \tag{16}$$

Combining (13) through (16) we thus get

$$\|f - s\| \leqslant [(3r + 2) + 4C]\,\omega(f, \delta)$$

and the proof is complete.

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